

GLUING SEMICLASSICAL RESOLVENT ESTIMATES VIA PROPAGATION OF SINGULARITIES

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ABSTRACT. We use semiclassical propagation of singularities to give a general method for gluing together resolvent estimates. As an application we prove estimates for the analytic continuation of the resolvent of a Schrödinger operator for certain asymptotically hyperbolic manifolds in the presence of trapping which is sufficiently mild in one of several senses. As a corollary we obtain local exponential decay for the wave propagator and local smoothing for the Schrödinger propagator.

1. INTRODUCTION

In this paper we give a general method for gluing semiclassical resolvent estimates. As an application we obtain the following theorem.

Theorem 1.1. *Let (X, g) be an even asymptotically hyperbolic Riemannian manifold. Let*

$$P = h^2 \Delta_g - 1, \quad 0 < h \leq h_0.$$

Suppose that the trapped set of X , i.e. the set of maximally extended geodesics which are precompact, is either normally hyperbolic in the sense of §5.3 or hyperbolic with negative topological pressure at $1/2$ (see §5.4). Then the cutoff resolvent $\chi(P - \lambda)^{-1}\chi$ continues analytically from $\{\operatorname{Im} \lambda > 0\}$ to $[-E, E] - i[0, \Gamma h]$ for every $E \in (0, 1)$ and $\chi \in C_0^\infty(X)$, where it obeys the bound

$$\|\chi(P - \lambda)^{-1}\chi\|_{L^2 \rightarrow L^2} \leq a(h).$$

Here $h^{-1} \lesssim a(h) \lesssim h^{-N}$ and $\Gamma > 0$ are both determined only by the trapped set. Moreover, for $E' \in [-E, E]$ we have for some $C > 0$ the following quantitative limiting absorption principle:

$$\|\chi(P - E' - i0)^{-1}\chi\|_{L^2 \rightarrow L^2} \leq C \log(1/h)h^{-1}.$$

By an *even asymptotically hyperbolic manifold* we mean that X is the interior of \overline{X} , a compact manifold with boundary, and

$$g = \frac{dx^2 + \tilde{g}}{x^2}, \text{ near } \partial \overline{X}.$$

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Here $x \in C^\infty(\overline{X})$ is a boundary defining function and \tilde{g} is a family of metrics on $\partial\overline{X}$, smooth up to $\partial\overline{X}$,¹ and even in x (see [Gui05, Definition 1.2] for a more invariant way to phrase this last condition). The assumptions on the trapped set will be discussed in more detail in §5, but for now we remark that it suffices to take X negatively curved with a trapped set consisting of a single closed geodesic. Moreover, the compact part of the manifold on which the trapping occurs can be replaced by a domain with several convex obstacles. See §6 for a stronger result.

As already stated our methods work in much greater generality. Let (X, g) be a complete Riemannian manifold, $P = h^2\Delta_g + V - 1$ a semiclassical Schrödinger operator, $V \in C^\infty(X; \mathbb{R})$ bounded, $h \in (0, 1)$. Then P is essentially self-adjoint, $R(\lambda) = (P - \lambda)^{-1}$ is holomorphic in $\{\lambda : \operatorname{Im} \lambda \neq 0\}$. Moreover, in this set one has uniform estimates on $R(\lambda) : L^2 \rightarrow L^2$ as $h \rightarrow 0$, namely $\|R(\lambda)\| \leq 1/|\operatorname{Im} \lambda|$.

On the other hand, as λ approaches the spectrum, $R(\lambda)$ is necessarily not uniformly bounded (even for a single h). However, in many settings, e.g. on asymptotically Euclidean or hyperbolic spaces with a suitable decay assumption on V , the resolvent extends continuously to the spectrum (perhaps away from some thresholds), although only as an operator on weighted L^2 -spaces. Indeed under more restrictive assumptions it continues meromorphically across the continuous spectrum, typically to a Riemann surface ramified at thresholds: see e.g. [Mel95] for a general discussion.

It is very useful in this setting to obtain semiclassical resolvent estimates, i.e. estimates as $h \rightarrow 0$, both at the spectrum of P and for the analytic continuation of the resolvent, $R(\lambda)$. By scaling, these imply high energy resolvent estimates for non-semiclassical Schrödinger operators, which in turn can be used, for instance, to describe wave propagation, or more precisely the decay of solutions of the wave equation: see §6 for examples of such applications in our setting. For this purpose the most relevant estimates are those in a strip near the real axis for the non-semiclassical problem (which gives exponential decay rates for the wave equation), which translates to estimates in an $\mathcal{O}(h)$ neighborhood of the real axis for semiclassical problems.

The best estimates one can expect (on appropriate weighted spaces) are $\mathcal{O}(h^{-1})$; this corresponds to the fact that this problem is semiclassically of real principal type. However, if there is trapping, i.e. some trajectories of the Hamilton flow (or, in case $V = 0$, geodesics) do not escape to infinity, the estimates can be significantly worse (exponentially large: see [Bur02]) even at the real axis. Nonetheless, if the trapping is mild, e.g. the trapped set is hyperbolic, then one has polynomial, $\mathcal{O}(h^{-N})$, bounds in certain settings, see the work of Nonnenmacher-Zworski [NoZw09a, NoZw09b], Petkov-Stoyanov [PeSt10], and Wunsch-Zworski [WuZw11]. Moreover, on the real axis (i.e. for $R(\lambda + i0)$ with $\lambda \in \mathbb{R}$) [NoZw09a, WuZw11] prove $\mathcal{O}(\log(1/h)h^{-1})$ bounds, which work of Bony-Burq-Ramond [BBR10] shows to be optimal when any trapping is present.

¹We can reduce a more general case to this one. Namely, it suffices to assume that \tilde{g} is a 2-cotensor which is a metric on $\partial\overline{X}$ when restricted to $\partial\overline{X}$: see [JoSá00, Proposition 2.1].

That $\mathcal{O}(h^{-1})$ bounds hold in strips for nontrapping asymptotically hyperbolic manifolds was proved by the second author in [Vas10] (see also [Vas11] and [MSV11]). This result, along with the accompanying propagation of singularities theorem, is what makes it possible to use our gluing method to prove Theorem 1.1; see §4.2 for a discussion of how to.

Typically, for the settings in which one can prove polynomial bounds in the presence of trapping, one considers particularly convenient models in which one alters the problem away from the trapped set, e.g. by adding a complex absorbing potential. The natural expectation is that if one can prove such bounds in a thus altered setting, one should also have the bounds if one alters the operator in a different non-trapping manner, e.g. by gluing in a Euclidean end or another non-trapping infinity. In spite of this widespread belief, no general prior results exist in this direction, though in some special cases this has been proved using partially microlocal techniques, e.g. in work of Christianson [Chr07, Chr08] on resolvent estimates where the trapping consists of a single hyperbolic orbit, and in the work of the first author [Dat09], combining the estimates of Nonnenmacher-Zworski [NoZw09a, NoZw09b] with the microlocal non-trapping asymptotically Euclidean estimates of the second author and Zworski [VaZw00] as well as the more delicate non-microlocal estimates of Burq [Bur02] and Cardoso-Vodev [CaVo02]. Another example is work of the first author [Dat10] using an adaptation of the method of complex scaling to glue in another class of asymptotically hyperbolic ends to the estimates of Sjöstrand-Zworski [SjZw07] and [NoZw09a, NoZw09b]. It is important to point out, however, that in the present paper we glue the resolvent estimates *directly*, without the need for any information on how they were obtained, so for instance we do not need to construct a global escape function, etc. In addition to the above listed references, Bruneau-Petkov [BrPe00] give a general method for deducing weighted resolvent estimates from cutoff resolvent estimates, but they require that the operators in the cutoff estimate and the weighted estimate be the same.

In this paper we show how one can achieve this gluing in general, in a robust manner. The key point is the following. One cannot simply use a partition of unity to combine the trapping model with a new ‘infinity’ because the problem is not semiclassically elliptic. Thus, semiclassical singularities (i.e. lack of decay as $h \rightarrow 0$) propagate along null bicharacteristics. However, under a convexity assumption on the gluing region, which holds for instance when gluing in asymptotically Euclidean or hyperbolic models, following these singularities microlocally allows us to show that an appropriate three-fold iteration of this construction, which takes into account the bicharacteristic flow, gives a parametrix with $\mathcal{O}(h^\infty)$ errors. This in turn allows us to show that the resolvent of the glued operator satisfies a polynomial estimate, and that on asymptotically Euclidean and hyperbolic manifolds the order of growth is given by that of the model operator for the trapped region.

We state our general assumptions and main result precisely in the next section, and prove the result in §3. In §4 we will show how our assumptions near infinity are satisfied for various asymptotically Euclidean and asymptotically hyperbolic manifolds. In §5 we show how our assumptions near the trapped set are satisfied for various types of hyperbolic trapping. In §6 we give applications: a more precise version of Theorem 1.1, exponential decay for the wave equation, and local smoothing for the Schrödinger propagator.

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2. MAIN THEOREM

Before stating the abstract assumptions of our main theorem, we review some terminology from semiclassical analysis. For $a \in C^\infty(T^*X)$ a symbol supported in a coordinate patch, $\psi \in C^\infty(X)$ compactly supported in the patch, $\psi \equiv 1$ on a neighborhood of the projection to X of the support of a , $\text{Op}(a)$ is a semiclassical quantization given in local coordinates by

$$\text{Op}(a)u(z) = \frac{1}{(2\pi h)^n} \int e^{iz\zeta/h} a(z, \zeta) \widehat{(\psi u)}(\zeta) d\zeta.$$

See, for example, [DiSj99, EvZw11] for more information. We also say that a family of functions $u = (u_h)_{h \in (0,1)}$ on X is polynomially bounded if $\|u\|_{L^2} \lesssim h^{-N}$ for some N . The semiclassical wave front set, $\text{WF}_h(u)$, is defined for polynomially bounded u as follows: for $q \in T^*X$, $q \notin \text{WF}_h(u)$ iff there exists $a \in C_0^\infty(T^*X)$ with $a(q) \neq 0$ such that $\text{Op}(a)u = \mathcal{O}(h^\infty)$ (in L^2). One can also extend the definition to $q \in S^*X$ (thought of as the cosphere bundle at fiber-infinity in T^*X) by considering $a \in C_0^\infty(\overline{T^*X})$, where $\overline{T^*X}$ is the fiber-radial compactification of T^*X with $a(q) \neq 0$ for $q \in S^*X$; then $\text{WF}_h(u) = \emptyset$ implies $u = \mathcal{O}(h^\infty)$ (in L^2).

Let \overline{X} be a compact manifold with boundary, g a complete metric on X (the interior of \overline{X}), and $P = h^2\Delta_g + V$ a semiclassical Schrödinger operator on X with $V \in C^\infty(X; \mathbb{R})$. We have no further explicit conditions on the potential V , although the dynamical assumption (2.1) and the assumptions on the resolvents below will in practice restrict the potentials the theorem applies to. Let x be a boundary defining function, and let

$$X_0 \stackrel{\text{def}}{=} \{0 < x < 4\}, \quad X_1 \stackrel{\text{def}}{=} \{x > 1\}.$$

Recall that a bicharacteristic of P is an integral curve in T^*X of the Hamiltonian vector field associated to the Hamiltonian function $p = |\xi|_g^2 + V(x)$, and that the energy of a bicharacteristic is the level set of p in which it lies. Suppose that the bicharacteristics γ of P (by this we always mean bicharacteristics at energy in some fixed range $[-E, E]$) satisfy the convexity assumption

$$\dot{x}(\gamma(t)) = 0 \Rightarrow \ddot{x}(\gamma(t)) < 0, \tag{2.1}$$

in X_0 . In particular this rules out the possibility of any trapped trajectory entering X_0 . There may be more complicated behavior, including trapping, in $X \setminus X_0$.

Remark 2.1. If x is a boundary defining function, f is a C^∞ function on $[0, \infty)$ with $f' > 0$, and x satisfies (2.1) then so does $f \circ x$. In particular the specific constants above (as well as intermediate constants used below) are chosen only for convenience, and can be replaced by arbitrary constants so long as the ordering of the constants is preserved.

Let P_0 and P_1 be model differential operators on X_0 and X_1 respectively:

$$P|_{X_0} = P_0|_{X_0}, \text{ and } P|_{X_1} = P_1|_{X_1}. \tag{2.2}$$

The spaces X'_j on which the operators P_j are globally defined can differ from X away from X_j , as the operators will always be multiplied by a smooth cutoff function to the appropriate X_j . Assume however that no bicharacteristic of P_1 leaves X_1 and then returns later, i.e. that

$$X_1 \text{ is bicharacteristically convex in } X'_1. \quad (2.3)$$

Note that we do not assume that the P_j are self-adjoint; this is useful in the applications in §§5–6. However, the condition (2.2) makes P_j formally self-adjoint on X_j and in particular has real semiclassical symbol on T^*X_j , as a result of which bicharacteristics of P_j are well defined on T^*X_j . Note that one difference from the (in some ways related) “black-box” approach of Sjöstrand-Zworski [SjZw91] is that for us X'_1 will typically not be a compact manifold.

Remark 2.2. In some applications we may wish to divide T^*X , rather than simply X , into two overlapping regions with a different model operator on each one. To do this it is enough to take x to be a more general function in $C^\infty(T^*X)$, rather than a boundary defining function for \bar{X} as we do here. In that case $\mathcal{O}(h^\infty)$ error terms appear in the formulas in (2.2), and in several other formulas below, but the construction is essentially the same. For simplicity of exposition we do not pursue this level of generality further below.

Let $\rho_0 \in C^\infty(X'_0)$, $\rho_1 \in C^\infty(X'_1)$ be bounded functions, referred to as *weights* (in typical applications they may be compactly supported, decaying at infinity, or constant) such that $\rho_0 = 1$ on $X_0 \cap X_1$ and $\rho_1 = 1$ on X_1 . Let $\rho \in C^\infty(X)$ have $\rho = \rho_0$ on X_0 and $\rho = 1$ otherwise.

Let $\rho R(\lambda)\rho$ denote the weighted resolvent $\rho(P - \lambda)^{-1}\rho$ in $\{\text{Im } \lambda > 0\}$ or its meromorphic continuation where this exists in $\{\text{Im } \lambda \leq 0\}$, and similarly $\rho_0 R_0(\lambda)\rho_0$ and $\rho_1 R_1(\lambda)\rho_1$ where those weighted resolvents exist. Suppose that $\tilde{h}_0 \in (0, 1)$ and the $\rho_j R_j(\lambda)\rho_j$ continue from $\{\text{Im } \lambda > 0\}$ to

$$\lambda \in D \subset [-E, E] - i[0, \Gamma h], \quad 0 < h < \tilde{h}_0,$$

(note that D need not be open, and may in particular consist of $[-E, E] + i0$), and in that region they obey

$$\|\rho_j R_j(\lambda)\rho_j\| \leq a_j(h, \lambda) \lesssim h^{-N}, \quad 0 < h < \tilde{h}_0, \quad (2.4)$$

for some $a_j(h) \geq h^{-1}$. We call a resolvent satisfying (2.4) *polynomially bounded*.

Definition 2.1. Let $\lambda \in D$ and $q \in T^*X_j$ be in the characteristic set of $P_j - \lambda$, that is the zero set of $p_j - \text{Re } \lambda$, where p_j is the semiclassical principal symbol of P_j . Let $\gamma_- : (-\infty, 0] \rightarrow T^*X'_j$ (or $\gamma_- : (-t_q, 0] \rightarrow T^*X'_j$ in case this is not defined for all time) be the backward P_j -bicharacteristic from q . We say that the resolvent $R_j(\lambda)$ is *semiclassically outgoing at q* if

$$u \in L^2_{\text{comp}}(X_j) \text{ polynomially bounded, } \text{WF}_h(u) \cap \gamma_- = \emptyset \quad (2.5)$$

implies that

$$q \notin \text{WF}_h(R_j(\lambda)u). \quad (2.6)$$

We say that the resolvent $R_j(\lambda)$ is *off-diagonally semiclassically outgoing at q* if (2.6) holds provided we add $q \notin T^*\text{supp } u$ to the hypotheses (2.5).

Remark 2.3. Since u is compactly supported, the definition involves only the cutoff resolvent.

In this paper we only need to make the assumption that the resolvents $R_j(\lambda)$ are off-diagonally semiclassically outgoing. However, for brevity, we use the term ‘semiclassically outgoing’ in place of ‘off-diagonally semiclassically outgoing’ throughout the paper.

A reason for making the weaker hypothesis of being off-diagonally semiclassically outgoing is that it is sometimes easier to check: typically the Schwartz kernel of $R_j(\lambda)$ is simpler away from the diagonal than at the diagonal (where it may be a semiclassical paired Lagrangian distribution), and the off-diagonal outgoing property follows easily from the oscillatory nature of the Schwartz kernel there; see the third paragraph of §4.

Our microlocal assumption is then that

- (0-OG) $R_0(\lambda)$ is semiclassically outgoing at all $q \in T^*(X_0 \cap X_1)$ (in the characteristic set of P_0),
- (1-OG) $R_1(\lambda)$ is semiclassically outgoing at all $q \in T^*(X_0 \cap X_1)$ (in the characteristic set of P_1) such that γ_- is disjoint from $T^*(X'_1 \setminus (X \setminus X_0)) = T^*(X'_1 \cap \{x > 4\})$, thus disjoint from any trapping in X_1 .

In fact, for $R_0(\lambda)$, it is sufficient to have the property in Definition 2.1 for $u \in L^2(X_0 \cap X_1)$ (i.e. u supported in $X_0 \cap X_1$).

The main result of the paper is the following general theorem.

Theorem 2.1. *Under the assumptions of this section, there exists $h_0 \in (0, 1)$ such that for $h < h_0$, $R(\lambda)$ continues analytically to D , where it obeys the bound*

$$\|\rho R(\lambda)\rho\| \leq Ch^2 a_0^2 a_1.$$

In particular, when $a_0 = C/h$, we find that $\rho R(\lambda)\rho$ obeys (up to constant factor) the same bound as $\rho_1 R_1(\lambda)\rho_1$, the model operator with infinity suppressed.

Remark 2.4. The only way we use convexity is to argue that no bicharacteristics of P go from $\{x > 2 + \varepsilon\}$ to $\{x < 2\}$ and back to $\{x > 2 + \varepsilon\}$ for some $\varepsilon > 0$. This is also fulfilled in some settings in which the stronger condition (2.1) does not hold, for example when X has cylindrical or asymptotically cylindrical ends. In particular, some mild concavity is allowed.

3. PROOF OF MAIN THEOREM

Let $\chi_1 \in C^\infty(\mathbb{R}; [0, 1])$ be such that $\chi_1 = 1$ near $\{x \geq 3\}$, and $\text{supp } \chi_1 \subset \{x > 2\}$, and let $\chi_0 = 1 - \chi_1$.

Define a right parametrix for P by

$$F \stackrel{\text{def}}{=} \chi_0(x-1)R_0(\lambda)\chi_0(x) + \chi_1(x+1)R_1(\lambda)\chi_1(x) \quad (3.1)$$

We then put

$$PF = \text{Id} + [P, \chi_0(x-1)]R_0(\lambda)\chi_0(x) + [P, \chi_1(x+1)]R_1(\lambda)\chi_1(x) \stackrel{\text{def}}{=} \text{Id} + A_0 + A_1.$$

This error is large in h due to semiclassical propagation of singularities: in general we have only $\|A_j \rho_j\|_{L^2 \rightarrow L^2} \leq Ch a_j(h)$, and $h a_j(h) \geq 1$, and thus $\|A_j \rho_j\|$ typically does not go to 0 with h . (Note that there is no need for a weight on the left of A_j in view of the support of the cutoffs.) However, using an iteration argument we can replace it by a small error. Observe that by disjointness of supports of $d\chi_0(\cdot - 1)$ and χ_0 , resp. $d\chi_1(\cdot + 1)$ and χ_1 , we have

$$A_0^2 = A_1^2 = 0, \quad (3.2)$$

while Lemma 3.1 below implies that

$$\|A_0 A_1\|_{L^2 \rightarrow L^2} = \mathcal{O}(h^\infty). \quad (3.3)$$

This is the step in which we exploit the semiclassical propagation of singularities (see Figure 1). Note that we have

$$A_0 A_1 = [P, \chi_0(x-1)]\rho_0 R_0(\lambda)\rho_0 [P, \chi_1(x+1)]\rho_1 R_1(\lambda)\rho_1 \chi_1(x),$$

that is to say, inserting weights ρ_0 and ρ_1 amounts to multiplying by 1 thanks to the cutoff functions χ_j which are present.

Lemma 3.1. *Suppose that $\varphi_1, \varphi_2, \varphi_3$ are compactly supported semiclassical differential operators (i.e. given in local coordinates by $\sum_\alpha a_\alpha(z) h^{|\alpha|} D_z^\alpha$ where the sum is over a finite set of multiindices α) with*

$$\text{supp } \varphi_1 \subset \{2 < x\}, \quad \text{supp } \varphi_2 \subset \{1 < x < 2\}, \quad \text{supp } \varphi_3 \subset \{3 < x < 4\}.$$

Then

$$\|\varphi_3 R_0(\lambda) \varphi_2 R_1(\lambda) \varphi_1\|_{L^2 \rightarrow L^2} = \mathcal{O}(h^\infty).$$

Before proving this lemma we show how (3.3) implies Theorem 2.1. We solve away the first error by writing, using (3.2)

$$P(F - F A_0) = \text{Id} + A_1 - A_0 A_0 - A_1 A_0 = \text{Id} + A_1 - A_1 A_0.$$

Similarly we have

$$P(F - F A_0 - F A_1) = \text{Id} - A_1 A_0 - A_0 A_1.$$

The last term is already $\mathcal{O}(h^\infty)$ by (3.3), but $A_1 A_0$ is not yet small. We thus repeat this process for $A_1 A_0$ to obtain

$$\begin{aligned} P(F - F A_0 - F A_1 + F A_1 A_0) &= \text{Id} - A_0 A_1 + A_0 A_1 A_0 + A_1 A_1 A_0 \\ &= \text{Id} - A_0 A_1 + A_0 A_1 A_0. \end{aligned}$$

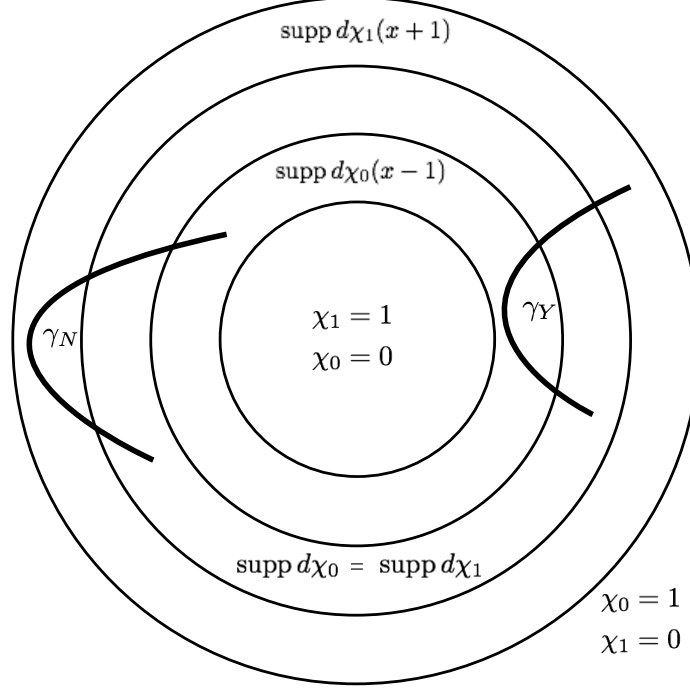


FIGURE 1. The concentric circles indicate integer level sets of x : the outermost one is $x = 1$ and the innermost $x = 4$. The supports of the various cutoffs are indicated (note that the supports are contained in the interiors of their respective annuli). The trajectory γ_N is ruled out by the convexity assumption (2.1), and this is exploited by Lemma 3.1. The trajectory γ_Y is possible, and this is the reason a third iteration is needed in the parametrix construction.

We now observe that both remaining error terms are of size $\mathcal{O}(h^\infty)$ thanks to (3.3). Correspondingly, $\text{Id} - A_0 A_1 + A_0 A_1 A_0$ is invertible for sufficiently small h , and the inverse is of the form $\text{Id} + E$, with $E = \mathcal{O}(h^\infty)$. To estimate the resolvent we write out

$$F - F A_0 - F A_1 + F A_1 A_0 = F - \chi_1(x+1)R_1(\lambda)\chi_1 A_0 + \chi_0(x-1)R_0(\lambda)\chi_0(-A_1 + A_1 A_0).$$

We then find that

$$\|\rho R(\lambda)\rho\| \leq C(a_0 + a_1 + 2ha_0 a_1 + h^2 a_0^2 a_1) \leq Ch^2 a_0^2 a_1.$$

This completes the proof that Lemma 3.1 implies Theorem 2.1. Note that only ρ_0 , the weight for R_0 , and not ρ_1 , appears in the definition of ρ . This is because $R_1(\lambda)$ is already multiplied by a compactly supported cutoff in every place where it appears in our parametrix (but this is not the case for $R_0(\lambda)$).

Lemma 3.1 follows from the following two lemmas, for the hypotheses (1)-(3) of Lemma 3.2 are satisfied by φ_j as in Lemma 3.1, and (4) follows from the support properties of φ_j and Lemma 3.3.

Lemma 3.2. *Suppose that $\varphi_1, \varphi_2, \varphi_3$ are semiclassical differential operators with the properties that*

- (1) φ_1 is supported in X_1 ,

- (2) φ_2, φ_3 are supported in $X_0 \cap X_1$,
- (3) $\text{supp } \varphi_3 \cap \text{supp } \varphi_2 = \emptyset$, and $\text{supp } \varphi_2 \cap \text{supp } \varphi_1 = \emptyset$,
- (4) there is no bicharacteristic of P_1 from a point $q_1 \in T^*(\text{supp } \varphi_1 \cup (X_1 \setminus X_0))$ to a point $q_2 \in T^* \text{supp } \varphi_2$ followed by a bicharacteristic of P_0 from q_2 to a point $q_3 \in T^* \text{supp } \varphi_3$.

Then

$$\|\varphi_3 R_0(\lambda) \varphi_2 R_1(\lambda) \varphi_1\|_{L^2 \rightarrow L^2} = \mathcal{O}(h^\infty),$$

Lemma 3.3. *There is no bicharacteristic of P_1 from a point $q_1 \in T^*\{x > 2\}$ to a point $q_2 \in T^*\{x < 2\}$ followed by a bicharacteristic of P_0 from q_2 to a point $q_3 \in T^*\{x > 2\}$.*

Proof of Lemma 3.3. We prove this first in the case where the two curves constitute a bicharacteristic of P . If there were such a bicharacteristic, say $\gamma: [t_0, t_1] \rightarrow T^*X$, with $x(\gamma(t_0)), x(\gamma(t_1)) > 2$, and $x(\gamma(\tau)) < \min(x(\gamma(t_0)), x(\gamma(t_1)))$ for some $\tau \in (t_0, t_1)$, then the function $x \circ \gamma$ would attain its minimum in the interior of (t_0, t_1) at some point (and would be < 2 there), and the second derivative would be nonnegative there, contradicting our convexity assumption (2.1).

We now reduce to this case. Assume that there are curves, $\gamma_0: [t_2, t_3] \rightarrow T^*X'_0$ a bicharacteristic of P_0 from q_2 to q_3 and $\gamma_1: [t_1, t_2] \rightarrow T^*X'_1$ a bicharacteristic of P_1 from q_1 to q_2 . Now, by the bicharacteristic convexity of X_1 in X'_1 , γ_1 is completely in X_1 (since its endpoints are there), so it is a P bicharacteristic. On the other hand, γ_0 need not be a P bicharacteristic since it might intersect $T^*(X_1 \setminus X_0)$. However, taking infimum t'_3 of times t at which $x(\gamma(t)) \geq x(q_3)$, $\gamma_0|_{[t_2, t'_3]}$ is a P bicharacteristic since it is disjoint from $T^*\{x > x(q_3)\}$ in view of $x(q_2) < 2$ and the intermediate value theorem. Thus, $\gamma: [t_1, t'_3] \rightarrow T^*X$ given by γ_1 on $[t_1, t_2]$ and γ_0 on $[t_2, t'_3]$ is a P bicharacteristic, with $x(\gamma(t_1)) > 2$, $x(\gamma(t_2)) < 2$, $x(\gamma(t'_3)) > 2$, completing the reduction to the case in the previous paragraph. \square

Proof of Lemma 3.2. First suppose that $u \in L^2(X)$ is polynomially bounded; we claim that

$$\|\varphi_3 R_0(\lambda) \varphi_2 R_1(\lambda) \varphi_1 u\|_{L^2} = \mathcal{O}(h^\infty). \quad (3.4)$$

For this, it suffices to show that $\text{WF}_h(\varphi_3 R_0(\lambda) \varphi_2 R_1(\lambda) \varphi_1 u) = \emptyset$. Note that by the polynomial boundedness assumption on the resolvent, $\varphi_3 R_0(\lambda) \varphi_2 R_1(\lambda) \varphi_1 u$, as well as $\varphi_2 R_1(\lambda) \varphi_1 u$, are polynomially bounded.

So suppose $q_3 \in \text{WF}_h(\varphi_3 R_0(\lambda) \varphi_2 R_1(\lambda) \varphi_1 u)$, so in particular $q_3 \in T^* \text{supp } \varphi_3 \cup S^* \text{supp } \varphi_3$ and, as φ_3 is microlocal, $q_3 \in \text{WF}_h(R_0(\lambda) \varphi_2 R_1(\lambda) \varphi_1 u)$. Now, if q_3 is not in the characteristic set of P_0 , then by microlocal ellipticity of P_0 , $q_3 \in \text{WF}_h(\varphi_2 R_1(\lambda) \varphi_1 u)$, thus in $T^* \text{supp } \varphi_2 \cup S^* \text{supp } \varphi_2$. This contradicts (3).

So we may assume that q_3 is in the characteristic set of P_0 (and hence in particular not in S^*X). By (0-OG), noting that φ_2 and φ_3 have disjoint supports, there is a point $q_2 \in \text{WF}_h(\varphi_2 R_1(\lambda) \varphi_1 u)$ on the backward P_0 -bicharacteristic from q_3 . Thus $q_2 \in T^* \text{supp } \varphi_2$ and $q_2 \in \text{WF}_h(R_1(\lambda) \varphi_1 u)$. By (1-OG), noting that φ_1 and φ_2 have disjoint supports, either the backward P_1 bicharacteristic from q_2 intersects $T^*(X_1 \setminus X_0)$, in which case we can take any

q_1 on it in this region, or there is a point q_1 on this backward bicharacteristic in $\text{WF}_h(\varphi_1 u)$, which is thus in $T^* \text{supp } \varphi_1$. Since this contradicts (4), it completes the proof of (3.4).

To complete the proof of the lemma, we just note that for any N , the family of operators

$$h^{-N} \varphi_3 R_0(\lambda) \varphi_2 R_1(\lambda) \varphi_1,$$

dependent on h and λ , is continuous on L^2 , and for each u , $h^{-N} \varphi_3 R_0(\lambda) \varphi_2 R_1(\lambda) \varphi_1 u$ is uniformly bounded in L^2 . Thus, by the theorem of Banach-Steinhaus, $h^{-N} \varphi_3 R_0(\lambda) \varphi_2 R_1(\lambda) \varphi_1$ is uniformly bounded (in h and λ) on L^2 , completing the proof of the lemma. \square

Remark 3.1. The application of Banach-Steinhaus is only needed because we merely made wavefront set assumptions in Definition 2.1. In practice, the wave front set statement is proved by means of a uniform estimate, and thus Banach-Steinhaus is superfluous.

Remark 3.2. Lemma 3.1 holds with the same proof if φ_j are instead semiclassical pseudo-differential operators with $\text{WF}_h \varphi_j$ in the cotangent bundle of the corresponding set. Note however that in this case slightly more care is needed in defining the φ_j since their Schwartz kernels may no longer be compactly supported. This could be useful for applications where P is not differential, as in [SjZw07].

4. MODEL OPERATORS NEAR INFINITY

In this section we describe some examples in which the assumptions on the model at infinity, P_0 , are satisfied. Recall that the assumptions on P_0 are of three kinds:

- (1) bicharacteristic convexity of level sets of x for $0 < x < 4$,
- (2) polynomial bounds for the cutoff resolvent,
- (3) semiclassically outgoing resolvent.

For simplicity, in this section we consider the case

$$P_0 = -h^2 \Delta_g - 1.$$

We start with some general remarks.

First, in the setting where X'_0 is diffeomorphic to \mathbb{R}^n , has nonpositive sectional curvature and, for fixed z_0 the function $x(z) = F(d(z, z_0))$ with $F' < 0$, (2.1) follows from the Hessian comparison theorem [ScYa94, VaWu05].

Next, the semiclassically outgoing assumption is satisfied for $R_0(\lambda)$ if the restriction of its Schwartz kernel to $(X_1 \cap X_0)^2 \setminus \text{Diag}$ is a semiclassical Fourier integral operator with canonical relation Λ' corresponding to forward propagation along bicharacteristics, i.e. $(y, z, \eta, \zeta) \in \Lambda'$ implies (y, η) is on the forward bicharacteristic segment from (z, ζ) . Here Diag is the diagonal in $(X_1 \cap X_0)^2$. Note that this is where restricting the semiclassical outgoing condition to its off-diagonal version is useful, in that usually the structure of the resolvent at the diagonal is slightly more complicated (though the condition would still hold); see also Remark 2.3.

4.1. Asymptotically Euclidean manifolds. If X is isometric outside of a compact set to Euclidean space we may take $X_0 = \mathbb{R}^n$ with the Euclidean metric g_0 , and x^{-1} the distance function from a point in \mathbb{R}^n . Thus, the convexity hypotheses (2.1) holds in view of geodesic convexity of the spheres. Moreover, for $\Gamma_1 > \Gamma > 0$, $\lambda_0 > 0$, the resolvent continues analytically to $\{\lambda : \operatorname{Im} \lambda < \Gamma h, \operatorname{Re} \lambda > \lambda_0\}$ as an operator

$$R(\lambda) : e^{-\Gamma_1|z|}L^2 \rightarrow e^{\Gamma_1|z|}L^2$$

with uniform estimates $\|R(\lambda)\| \leq Ch^{-1}$. Finally, $R(\lambda)$ is a semiclassical FIO associated to the forward flow; indeed, with $\sqrt{\cdot}$ the square root on $\mathbb{C} \setminus (-\infty, 0]$ which is positive for positive λ , its Schwartz kernel is (see e.g. [Mel95])

$$R(\lambda, y, z) = (h^{-1}\sqrt{\lambda})^{n-2} e^{i\sqrt{\lambda}|y-z|/h} a(\sqrt{\lambda}|y-z|/h),$$

where a is a symbol (away from the origin).

The applications in this case have already been treated in [NoZw09a, WuZw11], but for compactly supported cutoff functions. The novelty in the present paper in this setting is that we use exponential weights $e^{-\Gamma_1|z|}$. More general asymptotically Euclidean manifolds, whose metrics have holomorphic coefficients near infinity, could probably also be treated: see [WuZw00, WuZw11] for more details on the needed assumptions and the proof of the analytic continuation of the resolvent, and [VaZw00, Dat09] for semiclassical estimates and propagation of singularities.

4.2. Asymptotically hyperbolic manifolds. The convexity assumption (2.1) is satisfied for the geodesic flow on a general asymptotically hyperbolic metric. In the following lemma this is proved in a region $\{x < \varepsilon\}$, but a rescaling of the boundary defining function gives it in the region $\{x < 4\}$. The computation is standard, but we include it for the reader's convenience.

Lemma 4.1. *Let x be a boundary defining function on \overline{X} , a compact manifold with boundary, and let g be a metric on the interior of the form*

$$g = \frac{dx^2 + \tilde{g}}{x^2}, \text{ near } \partial\overline{X}$$

where \tilde{g} is a family of metrics on $\partial\overline{X}$, smooth up to $\partial\overline{X}$. Then for x sufficiently small we have

$$\dot{x}(t) = 0 \Rightarrow \ddot{x}(t) < 0$$

along geodesic bicharacteristics.

As remarked in the introduction, it is possible to reduce a more general form of the metric g to this one. Namely, it suffices to assume that \tilde{g} is a 2-cotensor which is a metric on $\partial\overline{X}$ when restricted to $\partial\overline{X}$: see [JoSá00, Proposition 2.1].

Proof. If (x, y) are coordinates on X near $\partial\overline{X}$ such that y are coordinates on $\partial\overline{X}$, and if ξ is dual to x and η to y , then the geodesic Hamiltonian is given by

$$|\zeta|^2 = \tau^2 + \tilde{g}^{-1}(\mu, \mu),$$

where $\tau = x\xi$ and $\mu = x\eta$, and \tilde{g}^{-1} is the bilinear form on $T^*\partial\overline{X}$ induced by \tilde{g} . Its Hamiltonian vector field is

$$H_{|\zeta|^2} = \partial_\xi |\zeta|^2 \partial_x - \partial_x |\zeta|^2 \partial_\xi + (\partial_\eta |\zeta|^2) \cdot \partial_y - (\partial_y |\zeta|^2) \cdot \partial_\eta.$$

We use $\partial_\xi = x\partial_\tau$, $\partial_\eta = x\partial_\mu$ and “ $\partial_x = \partial_x + x^{-1}\mu \cdot \partial_\mu + x^{-1}\tau\partial_\tau$ ”, where in the last formula the left hand side refers to (x, y, ξ, η) coordinates, and the right hand side to (x, y, τ, μ) coordinates. This gives

$$\begin{aligned} H_{|\zeta|^2} &= x\partial_\tau |\zeta|^2 (\partial_x + x^{-1}\mu \cdot \partial_\mu + x^{-1}\tau\partial_\tau) \\ &\quad - [(x\partial_x + \mu \cdot \partial_\mu + \tau\partial_\tau) |\zeta|^2] \partial_\tau + x(\partial_\mu |\zeta|^2) \cdot \partial_y - x(\partial_y |\zeta|^2) \cdot \partial_\mu. \end{aligned}$$

We cancel the $\partial_\tau (|\zeta|^2) \tau \partial_\tau$ terms, write $H_{\tilde{g}} = (\partial_\mu |\zeta|^2) \cdot \partial_y - (\partial_y |\zeta|^2) \cdot \partial_\mu$, substitute $|\zeta|^2 = \tau^2 + \tilde{g}(\mu, \mu)$, and use $\mu \cdot \partial_\mu \tilde{g}^{-1}(\mu, \mu) = 2\tilde{g}^{-1}(\mu, \mu)$. Now

$$H_{|\zeta|^2} = 2\tau x \partial_x + 2\tau \mu \cdot \partial_\mu - (2\tilde{g}^{-1}(\mu, \mu) + x\partial_x \tilde{g}^{-1}(\mu, \mu)) \partial_\tau + xH_{\tilde{g}}.$$

We now observe from this that, along flowlines of $H_{|\zeta|^2}$, we have $\dot{x} = 2\tau x$ and $\dot{\tau} = -2\tilde{g}^{-1}(\mu, \mu) - x\partial_x \tilde{g}^{-1}(\mu, \mu)$. Hence

$$\dot{x}(t) = 0 \Rightarrow \tau = 0,$$

in which case

$$\ddot{x} = -4x\tilde{g} - 2x^2\partial_x \tilde{g}^{-1}.$$

Since $\tilde{g}^{-1}|_{x=0}$ is positive definite, for sufficiently small x this is always negative. \square

If in addition \tilde{g} is even in x , in the sense that the Taylor series at $x = 0$ includes only even powers of x (or see [Gui05, Definition 1.2] for a more invariantly phrased version of this condition), then work of the second author [Vas10, Theorem 4.3], [Vas11, Theorem 5.1] implies the polynomial bound (2.4) and the outgoing condition (0-OG) for

$$P_0 = h^2 \Delta_g - 1,$$

when the manifold is nontrapping. The outgoing condition which is proved in those theorems, when restricted to data in $L^2(X_0 \cap X_1)$, is the same as that in condition (0-OG) (for this purpose the weights are irrelevant). The resolvent estimate [Vas11, (4.27)] is that with $\mathcal{R}(\sigma) = (\Delta_g - (\frac{n-1}{2})^2 - \sigma^2)^{-1}$ for $\text{Re } \sigma \gg 0$, for $s > \frac{1}{2} + \Gamma/2$ and $-\Gamma/2 < \text{Im } \sigma$,

$$\|x^{-(n-1)/2+i\sigma} \mathcal{R}(\sigma) f\|_{H_{|\sigma|^{-1}}^s(\overline{X}_{\text{even}})} \leq C|\sigma|^{-1} \|x^{-(n+3)/2+i\sigma} f\|_{H_{|\sigma|^{-1}}^{s-1}(\overline{X}_{\text{even}})}. \quad (4.1)$$

We will show that this implies

$$\|x^{1+\Gamma/2} R_0(\lambda) x^{5/2+\Gamma/2}\|_{L_g^2(X) \rightarrow L_g^2(X)} \leq C/h, \quad (4.2)$$

uniformly for $\text{Re } \lambda \in [-E, E]$, $\text{Im } \lambda > -\Gamma$. This argument is somewhat involved due to the rather different functions spaces appearing in (4.1) and (4.2), as already indicated by the presence of $\overline{X}_{\text{even}}$ in (4.1); the results of [Vas10, Theorem 4.3], [Vas11, Theorem 5.1] are obtained by extending an operator related to the spectral family of the Laplacian across $\partial\overline{X}_{\text{even}}$ to a larger space. Here $\overline{X}_{\text{even}}$ is \overline{X} as a topological manifold with boundary, but with smooth structure given by even (in x) smooth functions on \overline{X} ; effectively this means that the boundary defining function x is replaced by $\mu = x^2$.

Proof that (4.1) implies (4.2). We first recall the definition of $H_{|\sigma|^{-1}}^s(\overline{X}_{\text{even}})$, which is the standard semiclassical (with $|\sigma|^{-1}$ playing the role of the semiclassical parameter) Sobolev space on $\overline{X}_{\text{even}}$. A straightforward computation gives, see [Vas11, Section 1],

$$\|x^{-(n+1)/2}u\|_{L^2(\overline{X}_{\text{even}})} \sim \|u\|_{L_g^2(X)}, \quad (4.3)$$

where $L^2(\overline{X}_{\text{even}})$ is with respect to any smooth non-degenerate density on the compact manifold $\overline{X}_{\text{even}}$, while $L_g^2(X)$ is the metric L^2 -space. Furthermore, in local coordinates (μ, y) , using $2\partial_\mu = x^{-1}\partial_x$, for $l \geq 0$ integer, the squared high energy $H_{|\sigma|^{-1}}^l(\overline{X}_{\text{even}})$ norm of u is equivalent to

$$\sum_{k+|\alpha| \leq l} \| |\sigma|^{-k-|\alpha|} (x^{-1}\partial_x)^k \partial_y^\alpha u \|_{L^2(\overline{X}_{\text{even}})}^2.$$

We now convert (4.1) into an $H_0^s(\overline{X})$ -estimate, where $H_0^s(\overline{X})$ are the zero-Sobolev spaces of Mazzeo and Melrose [MaMe87], i.e. they are the Sobolev spaces measuring regularity with respect to $\text{Diff}_0(\overline{X})$, the algebra of differential operators generated by $\mathcal{V}_0(\overline{X})$, the Lie algebra of C^∞ vector fields vanishing at the boundary over $C^\infty(\overline{X})$, in the space $L_g^2(X)$. More precisely, we need the semiclassical version $H_{0,|\sigma|^{-1}}^s(\overline{X})$ of these spaces, in which $|\sigma|^{-1}$ times $\mathcal{V}_0(\overline{X})$ is used to generate the differential operators. The square high energy $H_{0,|\sigma|^{-1}}^l(\overline{X})$ norm of u is equivalent to

$$\sum_{k+|\alpha| \leq l} \| |\sigma|^{-k-|\alpha|} (x\partial_x)^k (x\partial_y)^\alpha u \|_{L_g^2(X)}^2.$$

Because of the ellipticity of Δ_g for these spaces (see [MSV11]), in the precise sense that the standard principal symbol is elliptic even in the semiclassical zero-calculus, this is also equivalent, when l is even, to

$$\|u\|_{L_g^2(X)}^2 + \| |\sigma|^{-l} \Delta_g^{l/2} u \|_{L_g^2(X)}^2. \quad (4.4)$$

This equivalence identifies the $H_{0,|\sigma|^{-1}}^l$ spaces with the usual semiclassical Sobolev spaces based on $L_g^2(X)$.

To make the conversion from (4.1) into an $H_{0,|\sigma|^{-1}}^s(\overline{X})$ -estimate, we remark that with $k+|\alpha| \leq l$,

$$(x^{-1}\partial_x)^k \partial_y^\alpha \in x^{-2k-|\alpha|} \text{Diff}_0^l(\overline{X}) \subset x^{-2l} \text{Diff}_0^l(\overline{X}),$$

and similarly for the high energy spaces, so

$$\|u\|_{H_{|\sigma|^{-1}}^l(\overline{X}_{\text{even}})} \lesssim \|x^{(n+1)/2-2l}u\|_{H_{0,|\sigma|^{-1}}^l(\overline{X})},$$

where the shift of $(n+1)/2$ in the exponent is due to the different normalization of the L^2 -spaces, (4.3). Thus, taking $s \geq 1$ integer, $s > 1/2 + \Gamma/2$, $\text{Im } \sigma > -\Gamma/2$, $\Gamma > 0$, and simply using $\|u\|_{L^2(\overline{X}_{\text{even}})} \leq \|u\|_{H_{|\sigma|^{-1}}^s(\overline{X}_{\text{even}})}$, we deduce that

$$\begin{aligned} \|x^{1-\text{Im } \sigma} \mathcal{R}(\sigma) f\|_{L_g^2(X)} &\leq C \|x^{-(n-1)/2+i\sigma} \mathcal{R}(\sigma) f\|_{L^2(\overline{X}_{\text{even}})} \leq C \|x^{-(n-1)/2+i\sigma} \mathcal{R}(\sigma) f\|_{H_{|\sigma|^{-1}}^s(\overline{X}_{\text{even}})} \\ &\leq C |\sigma|^{-1} \|x^{-(n+3)/2+i\sigma} f\|_{H_{|\sigma|^{-1}}^{s-1}(\overline{X}_{\text{even}})} \leq C |\sigma|^{-1} \|x^{-2s+1+i\sigma} f\|_{H_{0,|\sigma|^{-1}}^{s-1}(\overline{X})} \\ &\leq C |\sigma|^{-1} \|x^{-2s+1-\text{Im } \sigma} f\|_{H_{0,|\sigma|^{-1}}^{s-1}(\overline{X})}. \end{aligned}$$

Notice that there is a loss of x^{-2s} in the weight between the two sides. Although this simple argument does not give an optimal zero-Sobolev space estimate, to minimize losses take $3/2 + \Gamma/2 \geq s$, and $-\Gamma/2 < \text{Im } \sigma < -\Gamma/2 + 1/2$, so $-2s + 1 - \text{Im } \sigma > -5/2 - \Gamma/2$

$$\begin{aligned} \|x^{1+\Gamma/2}\mathcal{R}(\sigma)f\|_{L_g^2(X)} &\leq C\|x^{1-\text{Im } \sigma}\mathcal{R}(\sigma)f\|_{L_g^2(X)} \leq C|\sigma|^{-1}\|x^{-2s+1+i\sigma}f\|_{H_{0,|\sigma|^{-1}}^{s-1}(\overline{X})} \\ &\leq C|\sigma|^{-1}\|x^{-5/2-\Gamma/2}f\|_{H_{0,|\sigma|^{-1}}^{s-1}(\overline{X})}. \end{aligned} \quad (4.5)$$

Again using the ellipticity of Δ_g in the zero-calculus (as in (4.4)), allows one to strengthen the norm on the left hand side to $\|x^{1+\Gamma/2}\mathcal{R}(\sigma)f\|_{H_{0,|\sigma|^{-1}}^{s+1}(\overline{X})}$ – this simply requires a commutator argument or a parametrix with a smoothing (but not semiclassically trivial) error. For example, to strengthen the norm to $\|x^{1+\Gamma/2}\mathcal{R}(\sigma)f\|_{H_{0,|\sigma|^{-1}}^2(\overline{X})}$ we may write

$$\Delta_g x^{1+\Gamma/2}\mathcal{R}(\sigma)f = x^{1+\Gamma/2}f + \left(\frac{(n-1)^2}{4} + \sigma^2\right)x^{1+\Gamma/2}\mathcal{R}(\sigma)f + [\Delta_g, x^{1+\Gamma/2}]\mathcal{R}(\sigma)f. \quad (4.6)$$

Multiplying by $|\sigma|^{-2}$ and taking the $L_g^2(X)$ norm, we see that the first two terms are both controlled by the estimate (4.5), while the last is bounded by

$$|\sigma|^{-1}\|x^{1+\Gamma/2}\mathcal{R}(\sigma)f\|_{H_{0,|\sigma|^{-1}}^1(\overline{X})} \leq \frac{1}{2}\|x^{1+\Gamma/2}\mathcal{R}(\sigma)f\|_{H_{0,|\sigma|^{-1}}^2(\overline{X})}.$$

This implies that (4.5) holds with $L_g^2(X)$ norms replaced by $H_{0,|\sigma|^{-1}}^2$ norms, and iterating one can get as far as controlling the $H_{0,|\sigma|^{-1}}^{s+1}(\overline{X})$ norm (past which point the first term of (4.6) is no longer controlled).

To pass from estimates in $H_{0,|\sigma|^{-1}}^{s+1}(\overline{X})$ to estimates in $L_g^2(X)$ we use a similar procedure. For $K > 0$ fixed we have the semiclassical elliptic estimate

$$\|u\|_{H_{0,|\sigma|^{-1}}^{s-1}} \leq C|\sigma|^{-2}\|x^a(\Delta_g + K^2|\sigma|^2)x^{-a}u\|_{H_{0,|\sigma|^{-1}}^{s-3}}, \quad (4.7)$$

from which we deduce

$$\begin{aligned} &\|x^{1+\Gamma/2}\mathcal{R}(\sigma)f\|_{H_{0,|\sigma|^{-1}}^{s-1}} \\ &\leq \|x^{1+\Gamma/2}\mathcal{R}(iK|\sigma|)f\|_{H_{0,|\sigma|^{-1}}^{s-1}} + |\sigma|^2(1 + K^2)\|x^{1+\Gamma/2}\mathcal{R}(\sigma)\mathcal{R}(iK|\sigma|)f\|_{H_{0,|\sigma|^{-1}}^{s-1}} \\ &\leq C|\sigma|^{-2}\|x^{1+\Gamma/2}f\|_{H_{0,|\sigma|^{-1}}^{s-3}} + C|\sigma|\|x^{-5/2-\Gamma/2}\mathcal{R}(iK|\sigma|)f\|_{H_{0,|\sigma|^{-1}}^{s-1}} \\ &\leq C|\sigma|^{-1}\|x^{-5/2-\Gamma/2}f\|_{H_{0,|\sigma|^{-1}}^{s-3}}. \end{aligned}$$

Note that although the resolvent we use is $\mathcal{R}(\sigma) = (\Delta_g - (n-1)^2/4 - \sigma^2)^{-1}$, the shift by $(n-1)^2/4$ is not important when $|\sigma|$ is large and does not interfere with the application of (4.7). Iterating this we obtain the more standardly phrased weighted estimate

$$\|x^{1+\Gamma/2}\mathcal{R}(\sigma)f\|_{H_{0,|\sigma|^{-1}}^2(\overline{X})} \leq C|\sigma|^{-1}\|x^{-5/2-\Gamma/2}f\|_{L_g^2(X)},$$

which in turn implies (4.2). This is not optimal in terms of the weights which could be improved using $\overline{X}_{\text{even}}$ derivatives to estimate weights in the spirit of [Tay96, Chapter 4, Lemma 5.4], but this would result in a loss in terms of $|\sigma|$. \square

Another approach to obtaining polynomial boundedness of the resolvent and the semiclassical outgoing condition is possible in a special case. Let $X'_0 = \mathbb{B}^n$ with a metric which is asymptotically hyperbolic in the following stronger sense:

$$g_0 = g_{\mathbb{H}^n} + \chi_\delta(x)\tilde{g}, \quad P_0 = h^2 \left(\Delta_{g_0} + x^2 V_0 - \frac{(n-1)^2}{4} \right) - \lambda_0, \quad \lambda_0 > 0, \quad (4.8)$$

where $g_{\mathbb{H}^n}$ is the hyperbolic metric on \mathbb{B}^n and \tilde{g} is a smooth symmetric 2-cotensor on $\overline{\mathbb{B}^n}$, $V_0 \in C^\infty(\overline{\mathbb{B}^n})$, $\chi_\delta(t) = \chi(t/\delta)$, $\chi \in C^\infty(\mathbb{R})$ supported in $[0, 1)$, identically 1 near 0, and $\delta > 0$ sufficiently small. This is the setting considered by Melrose, Sá Barreto and the second author [MSV11]. Note that, although we have $g_0 = g_{\mathbb{H}^n}$ on a large compact set, the factor χ_δ does not change g_0 near infinity. Thus, after possibly scaling x , i.e. replacing it by x/ε , in the region $x < 4$ the cutoff $\chi_\delta \equiv 1$.

It is shown in [MSV11] that the Schwartz kernel of $(P_0 - \lambda)^{-1}$ is a semiclassical paired Lagrangian distribution, which is just a Lagrangian distribution away from the diagonal associated to the flow-out of the diagonal by the Hamilton vector field of the metric function, hence, as remarked at the beginning of the section, $(P_0 - \lambda)^{-1}$ is semiclassically outgoing. This also gives that $(P_0 - \lambda)^{-1}$ satisfies the bound in (2.4) with $D = [-E, E] - i[0, \Gamma h]$ and with $a_0 = C/h$ for arbitrary $\Gamma > 0, E \in (-\lambda_0, \lambda_0)$ with compactly supported cutoffs as a consequence of a semiclassical version of [GrUh90, Theorem 3.3]. Moreover, it is also shown in [MSV11] that the resolvent satisfies weaker polynomial bounds in weighted spaces, namely $R_j(\lambda) : x^a L^2 \rightarrow x^{-b} L^2$, $a, b > C$, with $a_0(h) = C'h^{-1-(n-1)/2}$. It is highly likely that the better bound $a_0 = C/h$ could be shown for the weighted spaces in this way as well; this could be proved by extending the approach of [GrUh90] in a manner that is uniform up to the boundary (i.e. infinity); this is expected to be relatively straightforward. The same results hold without modification in the case where X'_0 is a disjoint union of balls with g_0 and P_0 of the form (4.8) in each ball.

5. MODEL OPERATORS FOR THE TRAPPED SET

In this section we describe some examples in which the assumptions on the model near the trapped set, P_1 , are satisfied. The two main assumptions, polynomial boundedness of the resolvent (2.4) and the semiclassically outgoing property (1-OG), are the same as in the case of P_0 above, with the exception that the latter need only hold at points where the backward bicharacteristic is disjoint from any trapping in X_1 .

In §5.1 we prove that the semiclassically outgoing property (1-OG) holds for polynomially bounded resolvents when either a complex absorbing barrier is added near infinity (regardless of the cutoff or weight and regardless of the type of infinite end), and in §5.2 we prove it when infinity is Euclidean (with no complex absorption added) and the resolvent is polynomially bounded and suitably cutoff or weighted. In §5.3, §5.4, §5.5 we give examples of assumptions on the trapped set which imply that the resolvent is polynomially bounded.

5.1. Complex absorbing barriers. In this subsection we consider model operators of the form

$$P_1 = h^2 \Delta_g + V(x) - iW, \quad (5.1)$$

where $V \in C^\infty(X'_1, \mathbb{R})$ and $W \in C^\infty(X'_1; [0, 1])$ has $W = 0$ on X_1 and $W = 1$ off a compact set. Suppose that each backward bicharacteristic of $h^2 \Delta_g + V(x)$ at energy $\lambda \in [-E, E]$ enters either the interior of $T^*[(X_1 \setminus X_0) \cup W^{-1}(1)]$ in finite time. The strong assumptions on W remove the need for any further assumptions on V or on the metric.

The function W in (5.1) is called a complex absorbing barrier and serves to suppress the effects of infinity. In Lemma 5.1 we prove the needed semiclassical propagation of singularities in this setting, that is to say that $R_1(\lambda)$ is semiclassically outgoing in the sense of §2. After this, all that is needed to be in the setting of §2 is the convexity condition (2.1) and the resolvent estimate (2.4). In §5.3 and §5.4 we describe settings in which results of Wunsch-Zworski [WuZw11] and Nonnenmacher-Zworski [NoZw09a, NoZw09b] respectively give the needed bound (2.4).

For the following lemma we use a positive commutator argument based on an escape function as in [VaZw00], which is the semiclassical adaptation of the proof of [Hör71, Proposition 3.5.1]. The only slight subtlety comes from the interaction of the escape function with the complex absorbing barrier W and from the possibly unfavorable sign of $\text{Im } \lambda$, but the positive commutator with the self adjoint part of the operator overcomes these effects. See also [NoZw09a, Lemma A.2] for a similar result.

Lemma 5.1. *Suppose that P_1 is as in (5.1). Let $U \subset T^*(X'_1 \setminus (X \setminus X_0))$ be preserved by the backward bicharacteristic flow. If $u = u_h \in L^2(X_0 \cap X_1)$ has $\|u\|_{L^2} = 1$ and*

$$\|\text{Op}(a)(P_1 - \lambda)u\|_{L^2} = \mathcal{O}(h^\infty) \quad (5.2)$$

*for all $a \in C_0^\infty(T^*X'_1)$ with support in U and all $\lambda \in [-E, E] - i[0, \Gamma h]$, then for every $a \in C_0^\infty(T^*X'_1)$ with support in U we have also*

$$\|\text{Op}(a)u\|_{L^2} = \mathcal{O}(h^\infty). \quad (5.3)$$

The implicit constants in \mathcal{O} in (5.2) and (5.3) are uniform for $\lambda \in [-E, E] - i[0, \Gamma h]$.

Note that in view of Definition 2.1, this lemma implies assumption (0-OG)

Proof. In this proof all norms are L^2 norms. In the first step we use ellipticity to reduce to a neighborhood of the energy surface, and then a covering argument to reduce to a neighborhood of a single bicharacteristic segment. In the second step we construct an escape function (a monotonic function) along this segment. In the third step we implement the positive commutator method. Let

$$p = |\xi|_g^2 + V(x), \quad p_1 = p - iW.$$

Step 1. Observe first that for any $\delta > 0$, we can find $R_\delta(\lambda)$, a semiclassical elliptic inverse for P_1 on the set $\{|p_1 - \lambda| > \delta\}$, such that

$$\|\text{Op}(a)u\| = \|\text{Op}(a)R_\delta(\lambda)(P_1 - \lambda)u\| + \mathcal{O}(h^\infty)$$

as long as $\text{supp } a \subset \{|p_1 - \lambda| > \delta\}$. Since by the semiclassical composition formula the operator $\text{Op}(a)R_\delta(\lambda)$ is the quantization of a compactly supported symbol with support contained in $\text{supp } a$, plus an error of size $\mathcal{O}(h^\infty)$ (as an operator $L^2 \rightarrow L^2$), we have the lemma for a with $\text{supp } a \subset \{|p_1 - \lambda| > \delta\}$. It remains to study a with $\text{supp } a \subset \{|p_1 - \lambda| < 2\delta\}$. Note that this is a precompact set for δ small, because of the condition that $W = 1$ off of a compact set.

Now fix $a_0 \in C_0^\infty(T^*X_1)$ for which we wish to prove (5.3). Take U_0 with $\overline{U_0} \subset U$ such that U_0 is preserved by the backward flow and $\text{supp } a_0 \subset U_0$. For each $\zeta \in \overline{U_0} \cap p_1^{-1}(\lambda)$ put

$$T_\zeta \stackrel{\text{def}}{=} \sup\{t; t < 0, \Phi^t \zeta \in T^*W^{-1}([2\varepsilon, 1])\},$$

where Φ^t is the flow of the Hamiltonian vector field of p at time t , and $\varepsilon > 0$ will be specified later. The supremum is taken over a nonempty set, since each backward bicharacteristic of p was assumed to enter either $T^*(W^{-1}(1))$ or $T^*(X_1 \setminus X_0)$ in finite time, and the second possibility is ruled out by the assumption on U . We will prove the lemma for a which are supported in a sufficiently small neighborhood V_ζ of $\{\Phi^t \zeta; T_\zeta \leq t \leq 0\}$. This gives the full lemma because if δ small enough these neighborhoods together with $\{|p_1 - \lambda| > \delta\}$ cover all of $\text{supp } a_0$.

Step 2. To do this we take a tubular neighborhood $U_\zeta \subset U_0$ of $\{\Phi^t \zeta; T_\zeta \leq t \leq 0\}$, that is a neighborhood of the form

$$U_\zeta = \bigcup_{-\varepsilon_\zeta + T_\zeta < t < \varepsilon_\zeta} \Phi^t(\Sigma_\zeta \cap U_\zeta), \quad (5.4)$$

where $\Sigma_\zeta \subset T^*X$ is a hypersurface transversal to the bicharacteristic through ζ , and U_ζ and ε_ζ are small enough that

$$\bigcup_{-\varepsilon_\zeta + T_\zeta < t < T_\zeta + \varepsilon_\zeta} \Phi^t(\Sigma_\zeta \cap U_\zeta) \subset T^*W^{-1}([\varepsilon, 1]),$$

and also small enough that the map $U_\zeta \rightarrow (-\varepsilon_\zeta + T_\zeta, \varepsilon_\zeta) \times (\Sigma_\zeta \cap U_\zeta)$ defined by (5.4) is a diffeomorphism. We now use these ‘product coordinates’ to define an escape function as follows. Take

- $\varphi_\zeta \in C_0^\infty(\Sigma_\zeta \cap U_\zeta; [0, 1])$ with $\varphi_\zeta = 1$ near ζ , and
- $\chi_\zeta \in C_0^\infty((-\varepsilon_\zeta + T_\zeta, \varepsilon_\zeta); ([0, \infty)))$ with $\chi'_\zeta \leq -1$ near $[T_\zeta, 0]$ and $\chi'_\zeta \leq -2\Gamma\chi_\zeta$ on $[T_\zeta, \varepsilon_\zeta]$.

The constant Γ above is the same as the one in the statement of the lemma. Put

$$q_\zeta \stackrel{\text{def}}{=} \varphi_\zeta \chi_\zeta \in C_0^\infty(T^*X'_1), \quad \{p, q_\zeta\} = \varphi_\zeta \chi'_\zeta,$$

and let V_ζ be a neighborhood of $\{\Phi^t \zeta; T_\zeta \leq t \leq 0\}$ in which $\chi'_\zeta \leq -1$ and $\chi'_\zeta \leq -2\Gamma\chi_\zeta$. Take $b \geq 0$ such that

$$b^2 = -\{p, q_\zeta^2\} + r, \quad r \in C_0^\infty(T^*W^{-1}([\varepsilon, \infty))), \quad (5.5)$$

if necessary redefining χ_ζ so that b is smooth. By taking r large, we may ensure that

$$b^2 \geq 4\Gamma q_\zeta^2 \quad (5.6)$$

everywhere. Note that (5.6) follows from

$$-\{p, q_\zeta^2\} = -2\varphi_\zeta^2 \chi_\zeta \chi'_\zeta \geq 4\Gamma \varphi_\zeta^2 \chi_\zeta^2 = 4\Gamma q_\zeta^2$$

on V_ζ .

Step 3. Put $Q = \text{Op}(q_\zeta)$, $B = \text{Op}(b)$, $R = \text{Op}(r)$. Now

$$B^*B = \frac{i}{h}[Q^*Q, P] + R + hE,$$

where $P = h^2\Delta_g + V$ and where the error $E = \text{Op}(e)$ has $\text{supp } e \subset \text{supp } q_\zeta$. We have

$$\|Bu\|^2 = \frac{i}{h}\langle u, [Q^*Q, P]u \rangle + \langle u, Ru \rangle + h\langle u, Eu \rangle \leq \frac{i}{h}\langle u, [Q^*Q, P]u \rangle + h\|Eu\| + \mathcal{O}(h^\infty),$$

where we used $\|u\| = 1$ (which was an assumption) and $\|Ru\| = \mathcal{O}(h^\infty)$ (which follows from Step 1 above, since $R = \text{Op}(r)$ and $r \in C_0^\infty(T^*W^{-1}([\varepsilon, \infty)))$. Next

$$\begin{aligned} \frac{i}{h}\langle u, [Q^*Q, P]u \rangle &= -\frac{2}{h}\text{Im}\langle u, Q^*Q(P_1 - \lambda)u \rangle - \frac{2}{h}\text{Re}\langle u, Q^*QWu \rangle - \frac{2}{h}\langle u, Q^*Q\text{Im } \lambda u \rangle \\ &\leq -\frac{2}{h}\text{Re}\langle u, Q^*[Q, W]u \rangle - \frac{2}{h}\langle u, Q^*Q\text{Im } \lambda u \rangle + \mathcal{O}(h^\infty), \end{aligned}$$

where we used $\langle u, Q^*WQu \rangle \geq 0$ and $\|Q(P_1 - \lambda)u\| = \mathcal{O}(h^\infty)$ (see (5.2)). We will now show

$$-\text{Re}\langle u, Q^*([Q, W] + Q\text{Im } \lambda)u \rangle \leq \frac{h}{4}\|Bu\|^2 + \mathcal{O}(h^2)\|E'u\| + \mathcal{O}(h^\infty), \quad (5.7)$$

with $E' = \text{Op}(e')$ with $\text{supp } e' \subset \text{supp } q$. Then we will have

$$\|Bu\|^2 \leq \mathcal{O}(h)(\|Eu\| + \|E'u\|) + \mathcal{O}(h^\infty),$$

after which an iteration argument, for example as in [Dat09, Lemma 2], shows that $\|Bu\| = \mathcal{O}(h^\infty)$ allowing us to conclude. The iteration argument involves taking a nested sequence of escape functions q_ζ^j , with corresponding functions b^j as in (5.5) such that $\text{supp } q_\zeta^j$ is contained in the set where b^j is elliptic (bounded away from 0). This allows us to show that if $\|Q^j u\| \leq Ch^k\|u\|$, then $\|B^j u\| \leq Ch^{k+1/2}\|u\|$.

The estimate (5.7) is the slight subtlety discussed in the paragraph preceding the statement of the lemma. Because Q^* has real principal symbol of order 1, and $[Q, W]$ has imaginary principal symbol of order h , we have

$$|\text{Re}\langle u, Q^*[Q, W]u \rangle| = \mathcal{O}(h^2)\|E'u\| + \mathcal{O}(h^\infty),$$

with $E'' = \text{Op}(e'')$ with $\text{supp } e'' \subset \text{supp } q$. Meanwhile

$$\begin{aligned} \langle u, (B^*B + 4h^{-1}\text{Im } \lambda Q^*Q)u \rangle &\geq \langle u, (B^*B - 4\Gamma Q^*Q)u \rangle + \mathcal{O}(h^\infty) \\ &\geq -C'h\|E'''u\|^2 + \mathcal{O}(h^\infty), \end{aligned}$$

with $E''' = \text{Op}(e''')$ with $\text{supp } e''' \subset \text{supp } q$. For the second inequality we used the sharp Gårding inequality. Indeed, the semiclassical principal symbol of $B^*B - 4\Gamma Q^*Q$ is $b^2 - 4\Gamma q_\zeta^2$, and we may apply (5.6). \square

5.2. Euclidean ends. The model operator near the trapped set is of the form

$$P_1 = h^2 \Delta_g - 1$$

off of a compact set K' (which may contain X_1) and (X'_1, g) is isometric to Euclidean space there. Suppose that each backward bicharacteristic of $h^2 \Delta_g$ at energy $\lambda \in [-E, E]$ which enters $T^*(X_1 \cap X_0)$ also enters either $T^*(X_1 \setminus X_0)$ or $T^*(X'_1 \setminus K')$.

In this case, similarly to §4.1, the semiclassically outgoing condition, which is only needed in the Euclidean region (i.e. with backward bicharacteristic disjoint from K'), can be proved in several ways. One way is to use an escape function and positive commutator estimate as in Lemma 5.1: see [Dat09, Lemma 2] for a complete proof in a more general setting, based on the construction and estimates of [VaZw00]. Another way, which we only outline here, is to show the (off-diagonal) semiclassical FIO nature of $(P_1 - \lambda)^{-1}$ in this region, with Lagrangian given by the flow-out of the diagonal. But this follows from the usual parametrix identity, taking some $\chi \in C_0^\infty(X'_1)$ identically 1 on the compact set, using $G = (1 - \chi)\tilde{R}_0(\lambda)(1 - \chi)$ as the parametrix, with $\tilde{R}_0(\lambda)$ the Euclidean resolvent. Indeed, first for $\text{Im } \lambda > 0$, $(P_1 - \lambda)G = \text{Id} + E_R$, $G(P_1 - \lambda) = \text{Id} + E_L$, with E_R and E_L having Schwartz kernels with support in the left, resp. right factor in $\text{supp } \chi$ (e.g. $E_R = -\chi - [P_1, \chi]\tilde{R}_0(\lambda)$), so

$$(P_1 - \lambda)^{-1} = G - GE_R + E_L(P_1 - \lambda)^{-1}E_R;$$

this identity thus also holds for the analytic continuation. Now, even for the analytic continuation, G , E_L and E_R are semiclassical Lagrangian distributions away from the diagonal as follows from the explicit formula (where $\text{Im } \lambda$ is $\mathcal{O}(h)$), and if a point is in the image of the wave front relation of $G\chi_0$ or E_L (with χ_0 compactly supported, identically 1 on $\text{supp } \chi$) then it is on the forward bicharacteristic emanating from a point in $T^*\text{supp } \chi_0$, proving the semiclassically outgoing property of the second and third term of the parametrix identity.

5.3. Normally hyperbolic trapped sets. We take these conditions from [WuZw11]. Let (X'_1, g) be a manifold which is Euclidean outside of a compact set, let $V \in C_0^\infty(X_1, \mathbb{R})$, and let

$$P_1 = h^2 \Delta_g + V - 1 - iW,$$

with W as in (5.1) and $\text{supp } V \cap \text{supp } W = \emptyset$.

Define the backward/forward trapped sets by

$$\Gamma_\pm = \{\zeta \in T^*X_1 : \mp t \geq 0 \Rightarrow \Phi^t(\zeta) \notin \text{supp } W\},$$

where again $\Phi^t(\zeta) = \exp(tH_p)(\zeta)$. The trapped set is

$$K \stackrel{\text{def}}{=} \Gamma_+ \cap \Gamma_-.$$

We also define

$$\Gamma_\pm^\lambda = \Gamma_\pm \cap p^{-1}(\lambda), \quad K_\lambda = K \cap p^{-1}(\lambda).$$

Assume

- (1) There exists $\delta > 0$ such that $dp \neq 0$ on $p^{-1}((-\delta, \delta))$.
- (2) Γ_\pm are codimension one smooth manifolds intersecting transversely at K .

- (3) The flow is hyperbolic in the normal directions to K in $p^{-1}((-\delta, \delta))$: there exist subbundles E_λ^\pm of $T_{K_\lambda}(\Gamma_\pm^\lambda)$ such that

$$T_{K_\lambda}\Gamma_\pm^\lambda = TK_\lambda \oplus E_\lambda^\pm,$$

where

$$d\Phi^t: E_\lambda^\pm \rightarrow E_\lambda^\pm,$$

and there exists $\theta > 0$ such that for all $|\lambda| < \delta$,

$$\|d\Phi^t(v)\| \leq Ce^{-\theta|t|}\|v\| \text{ for all } v \in E_\lambda^\mp, \pm t \geq 0.$$

Here and below, by $d\Phi^s$ we mean the differential of $\Phi^s = \Phi^s(\zeta')$ as a function of ζ' .

This is the normal hyperbolicity assumption which we take from [WuZw11, §1.2]. This type of trapping appears in the setting of a slowly rotating Kerr black hole. Under these assumptions we have, from [WuZw11, (1.1)],

$$\|(P_1 - \lambda)^{-1}\|_{L^2(X'_1) \rightarrow L^2(X'_1)} \leq Ch^{-N},$$

for $\lambda \in [-E, E] - i[0, \Gamma h]$, for some $E, \Gamma, N > 0$. In particular, all the assumptions on P_1 and X'_1 in §2 are satisfied.

5.4. Trapped sets with negative topological pressure at 1/2. We take these conditions from [NoZw09a]. Let (X'_1, g) be a manifold which is Euclidean outside of a compact set, let $V \in C_0^\infty(X_1, \mathbb{R})$, and let

$$P_1 = h^2 \Delta_g + V - 1.$$

Let K_0 denote the set of maximally extended null-bicharacteristics of P_1 which are precompact. We assume that K_0 is hyperbolic in the sense that for any $\zeta \in K_0$, the tangent space to $p^{-1}(0)$ (the energy surface) at ζ splits into flow, unstable, and stable subspaces [KaHa95, Definition 17.4.1]:

- (1) $T_\zeta(p^{-1}(0)) = \mathbb{R}H_p(\zeta) \oplus E_\zeta^+ \oplus E_\zeta^-$, $\dim E_\zeta^\pm = \dim X - 1$.
- (2) $d\Phi^t(E_\zeta^\pm) = E_{\Phi^t(\zeta)}^\pm$, $\forall t \in \mathbb{R}$.
- (3) $\exists \theta > 0$, $\|d\Phi^t(v)\| \leq Ce^{-\theta|t|}\|v\|$, $\forall v \in E_\zeta^\mp, \pm t \geq 0$.

Here again $\Phi^t(\zeta) \stackrel{\text{def}}{=} \exp(tH_p)(\zeta)$. This condition is satisfied in the case where X is negatively curved near K_0 .

The unstable Jacobian $J_t^u(\zeta)$ for the flow at ζ is given by

$$J_t^u(\zeta) = \det \left(d\Phi^{-t}(\Phi^t(\zeta))|_{E_{\Phi^t(\zeta)}^+} \right).$$

We now define the topological pressure $P(s)$ of the flow on the trapped set, following [NoZw09a, §3.3] (see also [KaHa95, Definition 20.2.1]). We say that a set $S \subset K_0$ is (ε, t)

separated if, given any $\zeta_1, \zeta_2 \in S$, there exists $t' \in [0, t]$ such that the distance between $\Phi^{t'}(\zeta_1)$ and $\Phi^{t'}(\zeta_2)$ is at least ε . For any $s \in \mathbb{R}$ define

$$Z_t(\varepsilon, s) \stackrel{\text{def}}{=} \sup_S \sum_{\zeta \in S} (J_t^u(\zeta))^s,$$

where the supremum is taken over all sets $S \subset K_0$ which are (ε, t) separated. The pressure is then defined as

$$\mathcal{P}(s) = \lim_{\varepsilon \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t} \log Z_t(\varepsilon, s).$$

The crucial assumption on the dynamics of the bicharacteristic flow on the trapped set is that

$$\mathcal{P}(1/2) < 0.$$

Then from [NoZw09a, Theorem 3] and [NoZw09b, (1.5)] we have for any $\Gamma < |P(1/2)|$ and $\chi \in C_0^\infty(X'_1)$, there exist $C, E, N > 0$ such that

$$\|\chi(P_1 - \lambda)^{-1} \chi\|_{L^2 \rightarrow L^2} \leq Ch^{-1-N|\text{Im } \lambda|/h} \log(1/h),$$

for $\lambda \in [-E, E] - i[0, \Gamma h]$. In particular, all the assumptions on P_1 and X'_1 in §2 are satisfied.

5.5. Convex obstacles with negative abscissa of absolute convergence. We take these conditions from [PeSt10]. Let $(X'_1, g) = \mathbb{R}^n \setminus O$, where g is the Euclidean metric and where $O = O_1 \cup \dots \cup O_{k_0}$ is a union of disjoint convex bounded open sets with smooth boundary, and let

$$P_1 = h^2 \Delta_g - 1$$

with Dirichlet boundary conditions and. Assume that the O_j satisfy the no-eclipse condition: namely that for each pair $O_i \neq O_j$ the convex hull of $\overline{O_i}$ and $\overline{O_j}$ does not intersect any other $\overline{O_k}$.

In this setting having negative topological pressure at $1/2$ is equivalent to having negative abscissa of convergence of a certain dynamical zeta function, a condition under which a holomorphic continuation to strip of a polynomially bounded cutoff resolvent was first obtained by Ikawa [Ika88]. To define this, for γ a primitive periodic reflecting ray with m_γ reflections, let T_γ be the length of γ and P_γ the associated linear Poincaré map. Let $\lambda_{i,\gamma}$ for $i = 1, \dots, n-1$ be the eigenvalues of P_γ with $|\lambda_{i,\gamma}| > 1$. Let \mathcal{P} be the set of primitive periodic rays. Set

$$\delta_\gamma = -\frac{1}{2} \log(\lambda_{1,\gamma} \cdots \lambda_{n-1,\gamma}), \quad \gamma \in \mathcal{P}.$$

Let $r_\gamma = 0$ if m_γ is even and $r_\gamma = 1$ if m_γ is odd. The dynamical zeta function is given by

$$Z(s) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} (-1)^{mr_\gamma} e^{m(-sT_\gamma + \delta_\gamma)},$$

and the abscissa of convergence is the minimal $s_0 \in \mathbb{R}$ such that the series is absolutely convergent for $\text{Re } s > s_0$. Assume that

$$s_0 < 0.$$

For simplicity, assume in addition that $n = 2$. This assumption can be replaced by another which is weaker and more dynamical but also more complicated: see [PeSt10, Theorem 1.3] for a better statement. Then from [PeSt10, Theorem 1.3] we have for any $\chi \in C_0^\infty(X'_1)$

$$\|\chi(P_1 - \lambda)^{-1}\chi\|_{L^2 \rightarrow L^2} \leq Ch^{-N}$$

for $\lambda \in [-E, E] - i[0, \Gamma h]$, for some $N, E, m, C > 0$ and $\Gamma > |s_0|$. In particular, all the assumptions on P_1 and X'_1 in §2 are satisfied.

6. APPLICATIONS

We now give an improved version of Theorem 1.1.

Theorem 6.1. *Let (X, g) be even and asymptotically hyperbolic, let $V \in C_0^\infty(X; \mathbb{R})$, and let*

$$P = h^2 \Delta_g + V - 1, \quad p = |\xi|_g^2 + V - 1.$$

- (1) *Suppose for some $E > 0$ P has a normally hyperbolic trapped set on $p^{-1}[-E, E]$ in the sense of §5.3. Then there exist $h_0, N, \Gamma, C > 0$ such that*

$$\|x^{5/2+\Gamma/2} R(\lambda) x^{5/2+\Gamma/2}\|_{L^2 \rightarrow L^2} \leq Ch^{-N}$$

for $\lambda \in [-E, E] - i[0, \Gamma h]$ and $0 < h \leq h_0$.

- (2) *Suppose P has a hyperbolic trapped set on $p^{-1}(0)$ with $\mathcal{P}(1/2) < 0$ as in §5.4. Then for any $\Gamma < |\mathcal{P}(1/2)|$ there exist $E, h_0, N, C > 0$ such that*

$$\|x^{5/2+\Gamma/2} R(\lambda) x^{5/2+\Gamma/2}\|_{L^2 \rightarrow L^2} \leq Ch^{-N}$$

for $\lambda \in [-E, E] - i[0, \Gamma h]$ and $0 < h \leq h_0$.

- (3) *Let*

$$(\tilde{X}, g) = (\mathbb{R}^2, dr^2 + f(r)d\theta^2)$$

with $f \in C^\infty((0, \infty); (0, \infty))$ has $f(r) = r^2$ for r sufficiently small, $f(r) = \sinh^2(r)$ for r sufficiently large, and $f'(r) > 0$ for all r . Let $X = \tilde{X} \setminus O$ where O is a union of disjoint convex open sets all contained in the region where $f(r) = r^2$, satisfying the no-eclipse condition, with abscissa of convergence $s_0 < 0$ as in §5.5, and with Dirichlet boundary conditions imposed for $P = h^2 \Delta_g - 1$. Then there exist $E, h_0, N > 0$ and $\Gamma > |s_0|$ such that

$$\|x^{5/2+\Gamma/2} R(\lambda) x^{5/2+\Gamma/2}\|_{L^2 \rightarrow L^2} \leq Ch^{-N}$$

for $\lambda \in [-E, E] - i[0, \Gamma h]$ and $0 < h \leq h_0$.

Note that in case (3) we assume $f' > 0$ to rule out geometric trapping and to guarantee (2.1). Here $\exp(-(1+r^2)^{1/2})$, perhaps multiplied by a suitable large constant prefactor, plays the role of the boundary defining function x . The set X_1 encompasses the whole region where $f(r) \neq \sinh^2(r)$, and the set X_0 is contained in the region where $f(r) = \sinh^2(r)$.

The theorem follows immediately from the main theorem, Theorem 2.1, together with §4.2 (in which we show that the assumptions on P_0 are satisfied and derive the weights $x^{5/2+\Gamma/2}$),

and §5.3, resp. §5.4, resp. §5.5 in the cases (1), resp. (2), resp. (3) (in which we show that the assumptions on P_1 are satisfied).

Remark 6.1. The same results also hold when (X, g) has Euclidean ends in the sense of §4.1; one merely needs to use the results of §4.1 instead of those of §4.2. In this case the (mild) difference with previous authors is that we obtain the analytic continuation and the resolvent estimates for the resolvent with exponential weights (as in §4.1) rather than with compactly supported cutoff functions. Note however that in [BrPe00] Bruneau-Petkov give a method for passing from cutoff resolvent estimates to weighted resolvent estimates in such a situation.

In the special case where $V \equiv 0$ and

$$H = \Delta_g - \frac{(n-1)^2}{4},$$

we can obtain a resonant wave expansion as a corollary. Indeed, we have the resolvent estimate

$$\|x^{5/2+\Gamma/2}R(z)x^{5/2+\Gamma/2}\|_{L^2 \rightarrow L^2} \leq C|z|^{N-2}, \quad |\operatorname{Re} z| > z_0, \quad \operatorname{Im} z > -|\Gamma|/2, \quad (6.1)$$

where $R(z)$ is now $(H - z^2)^{-1}$ for $\operatorname{Im} z > 0$ or its meromorphic continuation for $\operatorname{Im} z < 0$. This follows from the substitution

$$h^2 z^2 = 1 + \lambda, \quad \operatorname{Re} z = h^{-1}.$$

On the other hand, work of Mazzeo-Melrose [MaMe87] and Guillarmou [Gui05] (see also [Vas10, Vas11]) shows that the weighted resolvent $x^{5/2+\Gamma/2}R(z)x^{5/2+\Gamma/2}$ continues meromorphically to $\{\operatorname{Im} z > -|\Gamma|/2\}$. From this the following resonant wave expansion follows.

Corollary 6.1. *Suppose u solves*

$$(\partial_t^2 + H)u = 0, \quad u|_{t=0} = f, \quad \partial_t u|_{t=0} = g \quad (6.2)$$

for $f, g \in C_0^\infty(X)$, with support disjoint from the convex obstacles in the case (3) above (and with no restriction on the support in cases (1) and (2)). Then

$$u(t) = \sum_{\operatorname{Im} z_j > -\Gamma/2} \sum_{m=0}^{M(z_j)} e^{-itz_j} t^m w_{z,j,m} + E(t, x). \quad (6.3)$$

The sum is taken over poles of $R(z)$, $M(z_j)$ is the algebraic multiplicity of the pole at z_j , and the $w_{z,j,m} \in C^\infty(X)$ are eigenstates or resonant states. The error term obeys the estimate

$$|\partial^\alpha E(t, x)| \leq C_{\alpha, \epsilon} e^{-t(\Gamma/2 - \epsilon)}$$

for every $\epsilon > 0$ and multiindex α , uniformly over compact subsets of X .

Note that the sum in (6.3) is finite, thanks to the resonance free strip established by the estimate (6.1). This is a standard consequence of the resolvent estimate and the meromorphic continuation, by taking a Fourier transform in time and then performing a contour deformation. See for example [Dat10, §6.3] for a similar result, and [MSV08, §4] for a similar result

with an asymptotic extending to infinity in space. We sketch the proof here: see [Dat10, §6.3] for more details. When $f \equiv 0$, we can write

$$u(t) = \frac{1}{2\pi} \int_{-\infty+iK}^{\infty+iK} e^{-izt} R(z) g \, dz,$$

where $K > (n-1)/2$. The proof then proceeds by contour deformation from $\{\operatorname{Im} z = K\}$ to $\{\operatorname{Im} z = -\Gamma/2\}$. The residues at the poles of the resolvent produce the terms of the expansion in (6.3), and the resolvent estimate

$$\|x^{5/2+\Gamma/2} R(z) x^{5/2+\Gamma/2}\|_{H^{s+N} \rightarrow H^s} \leq C |z|^{-2}, \quad (6.4)$$

for any s , justifies the deformation and controls the H^s norm of the error (on compact sets, or in suitably weighted spaces) in terms of the H^{s+N} norm of g . The estimate (6.4) can be derived from the $L^2 \rightarrow L^2$ estimate (6.1) following the same procedure as in §4.2 above. The case where $f \not\equiv 0$ and $g \equiv 0$ can be deduced similarly by differentiating the equation (6.2) in t , and then the general case follows by superposition of these two cases.

In many settings better resolvent estimates are available in the physical half plane $\operatorname{Im} \lambda > 0$. More specifically, we obtain the following theorem (see [WuZw11, (1.1)] and [NoZw09a, (1.17)] for the corresponding resolvent estimates for the trapping model operators).

Theorem 6.2. *Let (X, g) be even and asymptotically hyperbolic, let $V \in C_0^\infty(X; \mathbb{R})$, and let*

$$P = h^2 \Delta_g + V - 1.$$

Suppose P has a normally hyperbolic trapped set in the sense of §5.3 or a hyperbolic trapped set with $\mathcal{P}(1/2) < 0$ as in §5.4. Then for any $\chi \in C_0^\infty(X)$ there exist $E, h_0, C > 0$ such that

$$\|\chi R(\lambda) \chi\|_{L^2 \rightarrow L^2} \leq C \log(1/h) h^{-1}$$

for $\lambda \in [-E, E] + i[0, \infty)$, $0 < h \leq h_0$.

In [BBR10], Bony-Burq-Ramond prove that for P a semiclassical Schrödinger operator on \mathbb{R}^n , the presence of a single trapped trajectory implies that

$$\log(1/h) h^{-1} \leq C \sup_{\lambda \in [-\varepsilon, \varepsilon]} \|\chi R(\lambda) \chi\|,$$

provided $\chi \in C_0^\infty(X)$ is 1 on the projection of the trapped set. Consequently, in that setting (and probably in general), Theorem 6.2 is optimal.

From Theorem 6.2 it follows by a standard TT^* argument as in [Dat09, §6] that the Schrödinger propagator exhibits local smoothing with loss:

$$\int_0^T \|\chi e^{-it\Delta_g} u\|_{H^{1/2-\varepsilon}}^2 dt \leq C_{T,\varepsilon} \|u\|_{L^2}^2,$$

for any $T, \varepsilon > 0$. In fact, the main resolvent estimate of [Dat09] follows from Theorem 2.1 above, because the model operator near infinity, P_0 can be taken to be a nontrapping scattering Schrödinger operator, for which the necessary resolvent and propagation estimates were proved in [VaZw00]. Moreover, Burq-Guillarmou-Hassell [BGH10] show that when $\mathcal{P}(1/2) < 0$ semiclassical resolvent estimates with logarithmic loss can be used to deduce Strichartz estimates with no loss on a scattering manifold (a manifold with asymptotically

Euclidean or asymptotically conic ends in a sense which generalizes that of §4.1), and the same result probably holds on the asymptotically hyperbolic spaces considered here. See also [BGH10] for more references and a discussion of the history and of recent developments in local smoothing and Strichartz estimates.

Another possible application of the method is to give alternate proofs of cutoff resolvent estimates in the presence of trapping, where the support of the cutoff is disjoint from the trapping. As mentioned in the introduction, estimates of this type were proved by Burq [Bur02] and Cardoso and Vodev [CaVo02] and take the form

$$\|\chi R(\lambda)\chi\|_{L^2 \rightarrow L^2} \leq Ch^{-1},$$

for $\text{Im } \lambda > 0$, where $\chi \in C^\infty(X)$ vanishes on the convex hull of the trapped set and is either compactly supported or suitably decaying near infinity. Indeed a related method based on propagation of singularities was used in [DaVa10] to prove such a result.

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